Discrete Convexity in Probability, Tools & Applications

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Convex Sets

Convex Sets

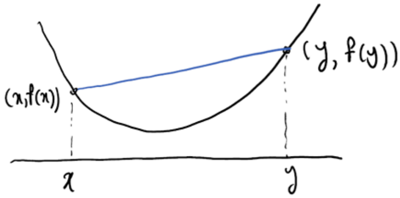
How to tell if a shape is convex?

A set in the Euclidean space is convex if it has "no holes" or "dents"

Definition 1 (convex sets).

A set $K \subseteq \mathbb{R}^n$ *is convex, if for any* $x, y \in K$ *and* $0 \leq \lambda \leq 1$,
$\lambda x + (1 - \lambda)\, y \in K$.

Convex Functions



Every line segment joining two points on its graph does not lie below the graph at any point

Definition 2 (convex functions).

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if its domain is a convex set and for all x, y in its domain, and $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1 - \lambda)\, y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

A Generalization...

- A convenient generalization of the standard convexity definition is the following:

Definition 3 ($s-$concave functions).

Fix $s \in [-\infty, \infty]$*. A function* $f : \mathbb{R}^n \to [0, \infty)$ *is* $s-$*concave if*

$$f((1-\lambda)x + \lambda y) \geq \left[(1-\lambda)\, f(x)^s + \lambda f(y)^s\right]^{1/s},$$

whenever $f(x)f(y) > 0$*.*

- The parameter s is understood as a *convexity parameter*.

s-concavity of Functions ctd.

Definition 4 ($s-$concave functions).

Fix $s \in [-\infty, \infty]$. A function $f : \mathbb{R}^n \to [0, \infty)$ is $s-$concave if

$$f((1-\lambda)x + \lambda y) \geq \left[(\mathbf{1}-\lambda)\,\mathbf{f}(\mathbf{x})^{\mathbf{s}} + \lambda \mathbf{f}(\mathbf{y})^{\mathbf{s}}\right]^{\mathbf{1/s}},$$

whenever $f(x)f(y) > 0$.

- If $s = +\infty$, then $f((1-\lambda)x + \lambda y) \geq \max\{f(x), f(y)\}$.
- If $s = 0$, then $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$.
- If $s = -\infty$, then $f((1-\lambda)x + \lambda y) \geq \min\{f(x), f(y)\}$.

Convexity of Measures

How to capture convexity of measures?

Definition 5 (Borell '75).

Fix $s \in [-\infty, \infty]$. *A finite measure* μ *on* $\mathbb{R}^n$ *is called* $s-$*concave if*

$$\mu((1-\lambda)A + \lambda B) \geq \left[(1-\lambda)\, \mu(A)^s + \lambda\, \mu(B)^s\right]^{1/s}$$

for non-empty Borel subsets $A, B \subseteq \mathbb{R}^n$.

- The case $s = -\infty$ describes the class of *convex measures* (or hyperbolic measures), defined as

$$\mu((1-\lambda)A + \lambda B) \geq \min\{\mu(A), \mu(B)\}.$$

Examples/ Motivation

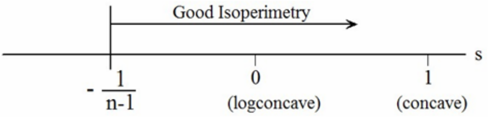
Which measures have convexity properties?

- Lebesgue measure on $\mathbb{R}^n$.
- Uniform measure on a convex body K in $\mathbb{R}^n$.
- Standard Gaussian measure on $\mathbb{R}^n$:

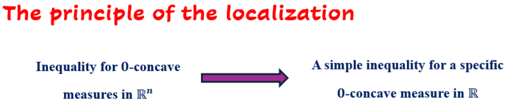
$$\gamma_n(x) = (2\pi)^{-n/2} e^{-\frac{\|x\|^2}{2}}.$$

Motivation for Convex Measures/Functions

- Extend general properties of log-concave measures (corresponds to $s = 0$) - concentration, isoperimetric inequality, etc.



- Generalize techniques like **localization** due to **Lovász** and **Simonovits '90s**.



Convexity in the Discrete Setting

Convexity in the discrete setting?

- A function $V : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ is said convex if

$$\Delta^2 V(z) := V(z-1) - 2V(z) + V(z+1) \geq 0 \quad \text{for all } z \in \mathbb{Z}.$$

- Equivalently, V is convex on $\mathbb{Z}$ if and only if there exists a continuous and convex function $\bar{V}$ such that $\bar{V} = V$ on $\mathbb{Z}$.

A natural extension of s-concavity in the discrete setting

Definition 6 (Discrete s-concave).

Fix $s \in [-\infty, \infty]$. A function $f : \mathbb{Z} \to \mathbb{R}^+$ is $s-$concave if $\{f > 0\}$ is an interval of integers and

$$f(k) \geq \left[\frac{f(k-1)^s + f(k+1)^s}{2} \right]^{1/s}.$$

- The case $s = 0$ corresponds to **discrete log-concavity (LC)**, i.e. $f^2(k) \geq f(k-1)\, f(k+1)$.

Log-concave Random Variables

Definition 7 (LC Random Variables).

A random variable X on $\mathbb{Z}$ is said to be **log-concave** *(w.r.t the counting measure) if its probability mass function $p(k) = \mathbb{P}(X = k)$ satisfies,*

$$p^2(k) \geq p(k-1)\,p(k+1) \ \text{ for all } k \in \mathbb{Z}\,.$$

Definition 8 (Generalized LC Random Variables).

A random variable X on $\mathbb{Z}$ is said to be **generalized log-concave** *w.r.t a reference measure γ, if its probability mass function p w.r.t γ is LC.*

- X is called **ultra-log-concave**, if γ is a Poisson measure.

Examples

Log-Concavity

Continuous Setting

Measures :-

- Lebesgue measure

Probability :-

- Normal
- Uniform
- Exponential
- Chi
- Laplace

Discrete Setting

- Bernoulli
- Binomial
- Poisson
- Geometric
- Negative binomial
- Hypergeometric

Motivation: Discrete Setting

Convex measures including log-concave measures and their geometry are well-understood in the continuous setting!!

One would like to investigate the discrete cases, at least for LC probabilities on $\mathbb{Z}$.

Example:

- Concentration behavior.
- Large and small deviation.
- Existence of moments.
- Stability under convolution.
- Geometric inequalities (Prékopa-Leindler etc.)
- Dilation inequalities.

An Optimization Technique?

- Concentration behavior
- Large and small deviation
- Existence of moments
- Stability under convolution
- Geometric inequalities
- Dilation inequalities

Constrained optimization-type problems

Goal: **Develop an optimization-type technique!**

Let's call this technique a "discrete localization"

A Discrete Localization

Notation: Let $a, b \in \mathbb{Z}$.

- $\mathcal{P}(\llbracket a, b \rrbracket)$: The set of all probabilities supported on $\llbracket a, b \rrbracket$.
- $h_1, h_2, ..., h_p$: Arbitrary real-valued functions defined on $\llbracket a, b \rrbracket$.
- $h = (h_1, h_2, ..., h_p)$.

Consider the following set:

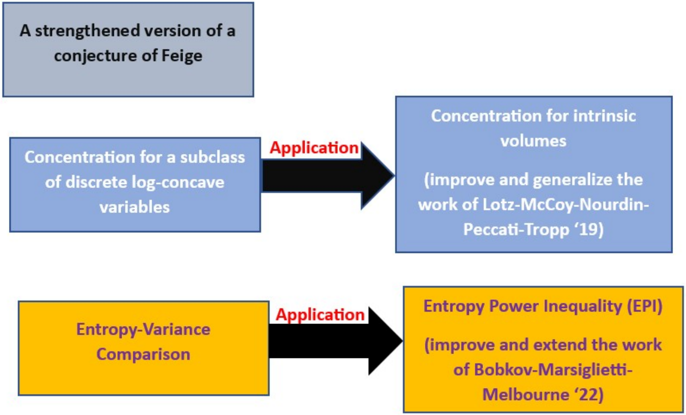
$$\mathcal{P}_h^{\gamma}(\llbracket a, b \rrbracket) = \{\mathbb{P}_X \in \mathcal{P}(\llbracket a, b \rrbracket) \ : \ \text{X log-concave}\,,\ \mathbb{E}[h_i(X)] \geq 0\}\,.$$

Theorem 1 (H. '24).

If $\mathbb{P}_X \in$ conv$(\mathcal{P}_h^{\gamma}(\llbracket a, b \rrbracket))$ is an extreme point, then it is **log piecewise affine** *(w.r.t γ).*

- This extends the localization result of Melbourne-Marsiglietti (2021).

Applications of Discrete Localization



Ultra Log-Concave Random Variables

Definition 9.

A random variable X taking values in $\{0, 1, 2, ...\}$ is said to be **ultra log-concave (ULC)** *if its probability mass function p is LC w.r.t Poisson measure, i.e.*

$$p^2(k) \geq \frac{k+1}{k} p(k+1)\, p(k-1) \ \textbf{for all } k \geq 1\,.$$

Examples:

- Binomial
- Poisson
- Sums of $i.i.d$ binomial with arbitrary parameters
- Hypergeometric distribution ($=$ sum of independent Bernoulli, **Ehm '91**).

Concentration For ULC Random Variables

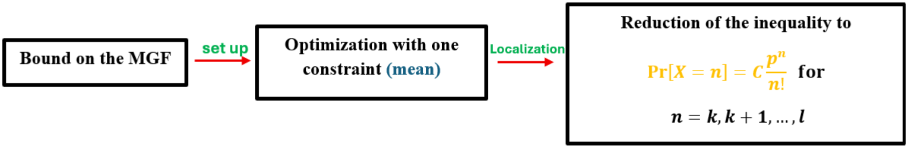
Theorem 2 (H., Marsiglietti, Melbourne '22).

For $X-$ ultra log-concave,

- $\mathbb{E}[e^{tX}] \leq \mathbb{E}[e^{tZ}]$ *for all* $t \in \mathbb{R}$, *where* $Z \sim Pois(\mathbb{E}[X])$.
- $\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq 2e^{\frac{-t^2}{2\,(t+\mathbb{E}[X])}}$ *for all* $t \geq 0$.

◇ In other words, all ultra log-concave sequences exhibit **Poisson-type concentration**.

Proof Ideas



- Fix an ultra log-concave random variable X_0. By approximation, assume that X_0 is compactly supported, say on $[\![k, l]\!]$.
- The idea is to use **the discrete localization** with the constraint function chosen as $h(z) = \mathbb{E}[X_0] - z$ for all $z \in [\![k, l]\!]$.
- Verify the inequality for extreme points, i.e. distributions of the form

$$\mathbb{P}[X = z] = C\frac{p^z}{z!} 1_{[\![k,l]\!]}(z), \;\; p, C > 0$$

- Conclude with the Cramér-Chernoff method.

A Consequence: Intrinsic Volumes

Corollary 1.

Let $K \subset \mathbb{R}^d$ be a non-empty convex body with intrinsic volume random variable Z_K. The variance satisfies,

$$\textit{Var}[Z_k] \leq d\,.$$

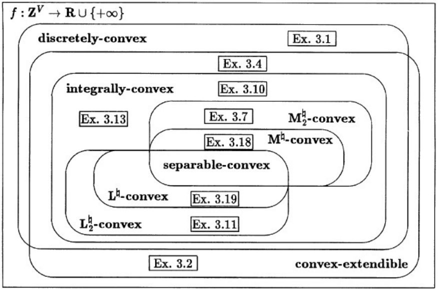
Moreover, in the range $0 \leq t \leq \sqrt{d}$,

$$\mathbb{P}(|Z_K - \mathbb{E}[Z_K]| \geq t\sqrt{d}) \leq 2e^{\frac{-t^2}{2}}.$$

- Improves upon a result of Lotz-McCoy-Nourdin-Peccati-Tropp (2019)

Future Directions

1. Investigate similar properties for (1-dimensional) discrete s-concave random variables.
2. Develop a localization for log-concave probabilities in $\mathbb{Z}^d$ and explore applications.



- Murota, Shioura. *Recent developments in discrete convex analysis.*

Thank You!