Discrete Convexity in Probability, Tools & Applications

Heshan Aravinda

Convex Sets

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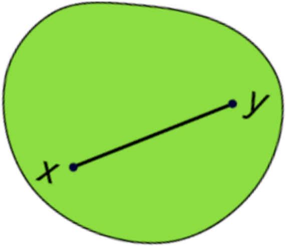
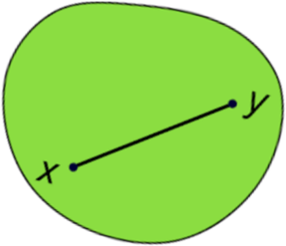
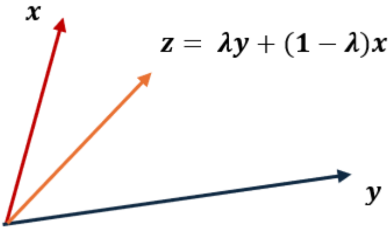


Convex Sets ctd.





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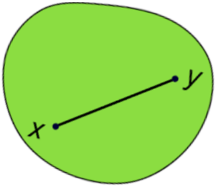
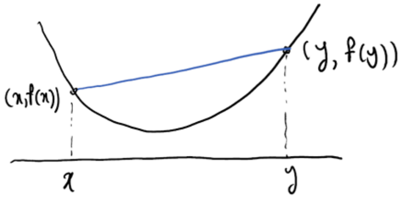


Definition 1 (convex sets).

A set $K \subseteq \mathbb{R}^n$ *is convex, if for any* $x, y \in K$ *and* $0 \leq \lambda \leq 1$,
$\lambda x + (1 - \lambda)\, y \in K$.

Convex Functions



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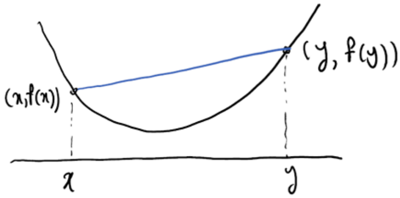

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Definition 2 (convex functions).

A function $f : \mathbb{R}^n \to \mathbb{R}$ *is convex if its domain is a convex set and for all* x, y *in its domain, and* $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)\, y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Convex/Concave Functions & Extensions

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Definition 3 ($s-$concave functions).

Fix $s \in [-\infty, \infty]$*. A function* $f : \mathbb{R}^n \to [0, \infty)$ *is* $s-$*concave if*

$$f((1-\lambda)x + \lambda y) \geq \left[(1-\lambda)\, f(x)^s + \lambda f(y)^s\right]^{1/s},$$

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- The parameter s is understood as a *convexity parameter*.

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NOTE: For $a, b > 0$, the map $s \to \left[(1-\lambda) a^s + \lambda b^s\right]^{1/s}$ is non-decreasing!!

Convexity of Measures

How to capture convexity of measures?

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Definition 5 (Borell '75).

Fix $s \in [-\infty, \infty]$. A finite measure μ on $\mathbb{R}^n$ is called $s-$concave if

$$\mu((1-\lambda)A + \lambda B) \geq [(1-\lambda)\,\mu(A)^s + \lambda\,\mu(B)^s]^{1/s}$$

for non-empty Borel subsets $A, B \subseteq \mathbb{R}^n$.

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for non-empty Borel subsets $A, B \subseteq \mathbb{R}^n$.

- The case $s = -\infty$ describes the largest class of measures, defined by

$$\mu((1-\lambda)A + \lambda B) \geq \min\{\mu(A), \mu(B)\}.$$

Its members are called *convex measures* (or hyperbolic measures).

Borell's Characterization of Convex Measures

- An absolutely continuous measure is log concave (corresponds to $s = 0$) if and only if its density is a log-concave function (**Prékopa**).

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Generalization of Prékopa's characterization

Theorem 1 (Borell '75).

A measure μ on $\mathbb{R}^n$ is κ- concave (for $\kappa \leq 1/n$) and absolutely continuous with respect to the Lebesgue measure if and only if it has a density that is a $s_{\kappa,n}$- concave function, where $s_{\kappa,n} = \dfrac{\kappa}{1 - \kappa n}$

Examples/ Motivation

Which measures have convexity properties?

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- Lebesgue measure on $\mathbb{R}^n$.

  (Brunn-Minkowski Inequality) If A and B are Borel subsets of $\mathbb{R}^n$, then

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  - ◇ $|\cdot|$ is $\frac{1}{n}$- concave.
  - ◇ $|\cdot|$ is 0- concave, i.e. log-concave.
  - ◇ $|\cdot|$ is $-\infty$- concave, i.e. convex.

Examples/ Motivation ctd.

- Uniform measure on a convex body K in $\mathbb{R}^n$ has density $|K|^{-1}\mathbb{1}_K$ which is ∞- concave, and thus the measure is $\frac{1}{n}$- concave.

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- Standard Gaussian measure on $\mathbb{R}^n$:

$$\gamma_n(x) = (2\pi)^{-n/2} e^{-\frac{\|x\|^2}{2}}.$$

is 0- concave, i.e. log-concave. Equivalently,

$$\gamma_n((1-\lambda)A + \lambda B) \geq \gamma_n(A)^{1-\lambda}\gamma_n(B)^{\lambda}.$$

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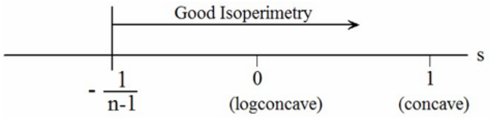
In addition, many probability distributions including Cauchy, Beta, Student's t, log- Normal, Pareto have stronger convexity properties.

Motivation for Convex Measures/Functions

- Extend general properties of log-concave measures (corresponds to $s = 0$) - concentration, isoperimetric inequality, etc.



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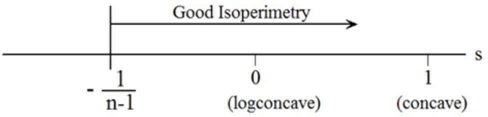

- Generalize techniques like **localization** due to **Lovász** and **Simonovits ’90s**.

The principle of the localization

Inequality for 0-concave measures in $\mathbb{R}^n$ ⟹ **A simple inequality for a specific 0-concave measure in $\mathbb{R}$**

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- A function $V : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ is said convex if

$$\Delta^2 V(z) := V(z-1) - 2V(z) + V(z+1) \geq 0 \text{ for all } z \in \mathbb{Z}.$$

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- Equivalently, V is convex on $\mathbb{Z}$ if and only if there exists a continuous and convex function $\bar{V}$ such that $\bar{V} = V$ on $\mathbb{Z}$.

A natural extension of s-concavity in the discrete setting

Definition 6 (Discrete s-concave).

Fix $s \in [-\infty, \infty]$. A function $f : \mathbb{Z} \to \mathbb{R}^+$ is $s-$concave if $\{f > 0\}$ is an interval of integers and

$$f(k) \geq \left[\frac{f(k-1)^s + f(k+1)^s}{2}\right]^{1/s}.$$

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- The cases $s \in \{-\infty, \infty\}$ are defined as limiting cases.
- The case $s = 0$ corresponds to **discrete log-concavity (LC)**, i.e. $f^2(k) \geq f(k-1)\, f(k+1)$.

Probability

- A discrete random variable X is called s-concave if its probability mass function (p.m.f) is s-concave w.r.t counting measure.

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Example 1.

For $s > 0$, the Zipf distribution given by the p.m.f

$$f(k) = \frac{1}{\zeta(s)} \frac{1}{k^{s+1}}, \; k = 1, 2, 3, \ldots$$

is $-\frac{1}{s+1}$- concave.

Log-concave Random Variables

Definition 7 (LC Random Variables).

A random variable X on $\mathbb{Z}$ is said to be **log-concave** *(w.r.t the counting measure) if its probability mass function $p(k) = \mathbb{P}(X = k)$ satisfies,*

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A random variable X on $\mathbb{Z}$ is said to be **generalized log-concave** *w.r.t a reference measure γ, if its probability mass function p w.r.t γ is LC.*

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Definition 8 (Generalized LC Random Variables).

A random variable X on $\mathbb{Z}$ is said to be **generalized log-concave** *w.r.t a reference measure γ, if its probability mass function p w.r.t γ is LC.*

- X is called **strongly LC** (or **ultra-log-concave**), if γ is a Poisson measure.

Examples

Log-Concavity

Continuous Setting

Measures :-

- Lebesgue measure

Probability :-

- Normal
- Uniform
- Exponential
- Chi
- Laplace

Discrete Setting

- Bernoulli
- Binomial
- Poisson
- Geometric
- Negative binomial
- Hypergeometric

Motivation: Discrete Setting

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One would like to investigate the discrete cases, at least for LC probabilities on $\mathbb{Z}$.

Example:

- Concentration behavior.
- Large and small deviation.
- Existence of moments.
- Stability under convolution.
- Geometric inequalities (Prékopa-Leindler etc.)
- Dilation inequalities.

An Optimization Technique?

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Constrained optimization-type problems

Goal: Develop an optimization-type technique!

Let's call this technique a "discrete localization"

A Discrete Localization

Notation: Let $a, b \in \mathbb{Z}$.

- $[\![a, b]\!] = \{a, a+1, a+2, ..., b\}$.
- $\mathcal{P}([\![a, b]\!])$: The set of all probabilities supported on $[\![a, b]\!]$.
- $h_1, h_2, ..., h_p$: Arbitrary real-valued functions defined on $[\![a, b]\!]$.
- $h = (h_1, h_2, ..., h_p)$.

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Consider the following set:

$$\mathcal{P}_h([\![a, b]\!]) = \{\mathbb{P}_X \in \mathcal{P}([\![a, b]\!]) \ : \ \text{X log-concave}\,,\ \mathbb{E}[h_i(X)] \geq 0\}\,.$$

A Discrete Localization ctd.

Theorem 2 (H. '22).

If $\mathbb{P}_X \in$ *conv*$(\mathcal{P}_h([\![a, b]\!]))$ *is an extreme point, then it is log piecewise affine.* **(⋆)**

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- More specifically, if e^{-V} is the probability mass function of $\mathbb{P}_X$, then V is the maximum of at most p discrete affine functions.

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Corollary 1 (Finite dimensional Krein-Milman).

Let $\Phi : \mathcal{P}_h([\![a,b]\!]) \to \mathbb{R}$ be convex. Then,

$$\sup_{\mathbb{P}_X \in \mathcal{P}_h([\![a,b]\!])} \Phi(\mathbb{P}_X) \le \sup_{\mathbb{P}_X \in \mathcal{A}_h([\![a,b]\!])} \Phi(\mathbb{P}_X),$$

where $\mathcal{A}_h([\![a,b]\!]) = \mathcal{P}_h([\![a,b]\!]) \cap \{\mathbb{P}_X : X$ with PMF as in $(\star)\}$

Applications

- Concentration behavior
- Large and small deviation
- Existence of moments
- Stability under convolution
- Geometric inequalities
- Dilation inequalities

CONCENTRATION FOR ULC

Ultra Log-Concave Random Variables

Definition 9 (ULC/ Strongly Log-Concave).

A random variable X taking values in $\{0, 1, 2, ...\}$ is said to be **ultra log-concave (ULC)** *if its probability mass function p is LC w.r.t Poisson measure, i.e.*

$$p^2(k) \geq \frac{k+1}{k}\, p(k+1)\, p(k-1) \ \ \textbf{for all } k \geq 1\,.$$

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Examples:

- Binomial
- Poisson
- Sums of $i.i.d$ binomial with arbitrary parameters
- Hypergeometric distribution ($=$ sum of independent Bernoulli, **Ehm '91**).

If X is strongly LC (or ultra-log-concave), then, how does X deviate from $\mathbb{E}[X]$?

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i.e,

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq D(t)$$

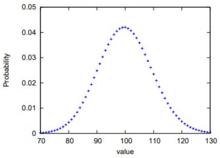

FIGURE 5. The Binomial distribution $B(1000, 0.1)$.

Concentration Phenomena

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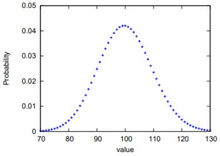

FIGURE 5. The Binomial distribution $B(1000, 0.1)$.

What does $D(t)$ look like?

Concentration For ULC Random Variables

Theorem 3 (H., Marsiglietti, Melbourne '22).

For any $X-$ ultra log-concave,

- $\mathbb{E}[e^{tX}] \leq \mathbb{E}[e^{tZ}]$ *for all* $t \in \mathbb{R}$, *where* $Z \sim Pois(\mathbb{E}[X])$.
- $\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq 2e^{\frac{-t^2}{2\,(t+\mathbb{E}[X])}}$ *for all* $t \geq 0$.

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◇ In other words, all ultra log-concave sequences exhibit **Poisson-type concentration**.

FEIGE'S CONJECTURE

Feige's Conjecture

Conjecture 1 (Feige '05).

Given n independent non-negative random variables $X_1, X_2, ..., X_n$ such that $\mathbb{E}[X_i] \leq 1$. Let $X = \sum_{i=1}^{n} X_i$. Then

$$\mathbb{P}(X < \mathbb{E}[X] + 1) \geq \frac{1}{e}.$$

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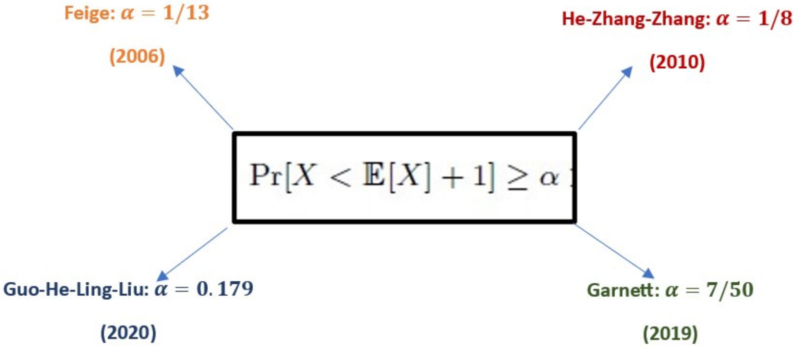
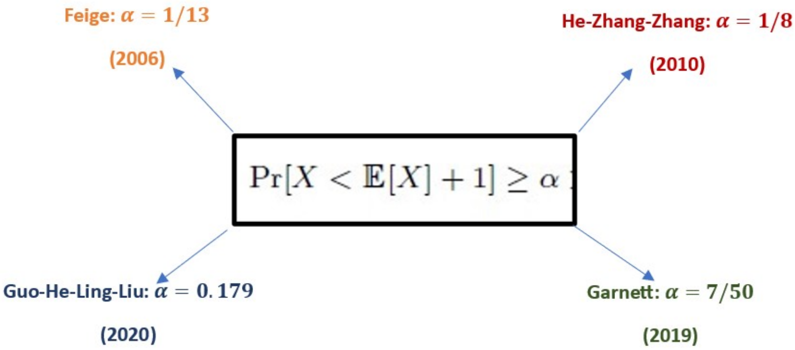
Let X_i be i.i.d, $X_i = n + 1$ with probability $\frac{1}{n+1}$ and $X_i = 0$ otherwise.

Then,

$$\mathbb{P}(\sum_{i=1}^{n} X_i < n + 1) = \mathbb{P}(\sum_{i=1}^{n} X_i = 0) = \left(1 - \frac{1}{n+1}\right)^n \to \frac{1}{e}$$


Feige: $\alpha = 1/13$
(2006)
He-Zhang-Zhang: $\alpha = 1/8$
(2010)
$\Pr[X < \mathbb{E}[X] + 1] \geq \alpha$
Guo-He-Ling-Liu: $\alpha = 0.179$
(2020)
Garnett: $\alpha = 7/50$
(2019)

What Is Known?



- The conjectured bound holds for binomial and independent Bernoulli sums (follows from a special case of **Samuel's conjecture**).
- For X-Poisson, $\mathbb{P}(X \leq \mathbb{E}[X]) > 1/e$ (**Teicher '55**).

Feige's Conjecture for LC Random Variables

Theorem 4 (Alqasem, H., Marsigliettí, Melbourne '24).

Let X *be a discrete log-concave random variable. Then*

$$\mathbb{P}(X < \mathbb{E}[X] + 1) \geq e^{-1}.$$

The inequality is sharp for distributions of the form $\mathbb{P}(X = k) = Cn^{k/n}$*, for* $k = 1, 2, ..., n$*.*

Proof Ideas

- By approximation, reduce the inequality to LC random variables with finite support.
- We can further assume that the support is $[\![1, N]\!]$.
- Let X_0 be any random variable supported on $\mathcal{P}([\![1, N]\!])$.
- Let $h : [\![1, N]\!] \to \mathbb{R}$ be given by $h(k) = \mathbb{E}[X_0] - k$.

$$\mathcal{P}_h([\![1, N]\!]) = \{\mathbb{P}_X \in \mathcal{P}([\![1, N]\!]) \,:\, \text{X log-concave}\,,\, \mathbb{E}[h(X)] \geq 0\}\,.$$

- The discrete localization (with a single constraint) implies the extreme points are log-affine, i.e. a random variable X with the probability mass function defined as,

$$p(n) = C\,\lambda^n \mathbb{1}_{[\![K,M]\!]}(n),\ \text{where}\,\lambda, C > 0\,\text{and}\ [\![K, M]\!] \subset [\![1, N]\!]$$

Proof Ideas ctd.

- Now, invoke Krein-Milman.
- Choose the convex functional $\Phi(\mathbb{P}_X) = \mathbb{P}_X(A)$, where A is a Borel subset in $\mathbb{R}$.
- In fact, take $A = [\mathbb{E}[X] + 1, \infty]$, so that

$$\Phi(\mathbb{P}_X) = \mathbb{P}_X(A) = \mathbb{P}(X \geq \mathbb{E}[X] + 1).$$

- We conclude by verifying this inequality $\mathbb{P}(X \geq \mathbb{E}[X] + 1) \leq 1 - \frac{1}{e}$ for X- log-affine.

Thank You!